MASTER LEVEL INTERNSHIP PROPOSAL: QUANTUM TRAJECTORIES: BEYOND THE UNIQUENESS OF THE INVARIANT MEASURE

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We first recall the definition of a projective space. Let $\sim$ be an equivalence relation on $\mathbb{C}^k \setminus \{0\}$ defined by $x \sim y$ iff there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $x = \lambda y$. Then the projective space $\mathrm{P}(\mathbb{C}^k)$ is the set of equivalent classes of $\sim$. We denote $\hat{x} \in \mathrm{P}(\mathbb{C}^k)$ the equivalent class of $x \in \mathbb{C}^k \setminus \{0\}$ and given $\hat{x} \in \mathrm{P}(\mathbb{C}^k)$, $x \in \mathbb{C}^k \setminus \{0\}$ is an arbitrary ℓ^2-norm one element of $\hat{x}$. To fix ideas, $\mathrm{P}(\mathbb{C}^k)$ is isometric to $S^{k-1}/U(1)$ where S^{k-1} is the k-dimensional complex ℓ^2-sphere.

Given a finite set of $k \times k$ complex matrices $\{V_j\}_{j=1}^{\ell}$ such that $\sum_j V_j^* V_j = \mathrm{Id}$, one can define a Markov process on the projective space $\mathrm{P}(\mathbb{C}^k)$ in the following way: Given state $\hat{x}_n \in \mathrm{P}(\mathbb{C}^k)$ at time $n \in \mathbb{N}$,

$$\hat{x}_{n+1} = \widehat{V_j x_n} \quad \text{with probability } \|V_j x_n\|_2^2.$$

Hence, the state at time $n+1$ depends on the state at time n and the randomly chosen matrix V_j. The kernel of this Markov process is then given for any continuous function $f : \mathrm{P}(\mathbb{C}^k) \to \mathbb{C}$ by

$$\Pi f(\hat{x}) = \sum_j f(\widehat{V_j x}) \|V_j x\|_2^2.$$

These processes can be viewed as iterated function systems and are therefore quite singular. For example, for appropriate choices of matrices, their invariant measure is supported on a set of non trivial fractal dimension. For that reason the proof of uniqueness of their invariant measure has long remained an open problem. However, recently, with two other collaborators we used a new estimation technique allowing us to solve this problem. [1] This proof opens a new door to the study of these processes that are central to the modeling of quantum optics experiments. Hence their name: Quantum trajectories. It also offers a new path to study some relevant questions concerning iid random products of matrices.

In this internship, we propose to study and eventually improve the proof of [1] with the aim of deriving a law of large numbers, a central limit theorem and a law of iterated logarithm for these processes. The idea is to adapt standard techniques developed for more regular Markov chains to our method of proof.

Depending on the student interests several other questions on these processes could be explored, from specific examples to large deviation principles or regularizations of these processes.

This work could eventually be continued during a Ph.D. Thesis in probability.

References

[1] T. Benoist, M. Fraas, Y. Pautrat and C. Pellegrini, *Invariant measure for quantum trajectories*, Probab. Theory Relat. Fields **174** (2019) 307–334

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Key words and phrases. Stochastic processes, Central limit theorem, Law of iterrated logarithm, large deviation principle.