MASTER LEVEL INTERNSHIP PROPOSAL: RÉNYI ENTROPY OF PRODUCT OF MATRICES MEASURES

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In this internship we propose to study the regularity of Rényi entropy for measures defined by products of matrices.

Given a set $\{V_j\}_{j=1}^{\ell}$ of $d\times d$ complex matrices such that $\sum_j V_j^*V_j = \mathrm{Id}$, one can define a probability measure on the sequences of elements of $\{1,\dots,\ell\}$, $\Omega = \{1,\dots,\ell\}^{\mathbb{N}}$ by setting

$$\mathbb{P}(C_{j_1,\dots,j_n}) = \frac{1}{d}\mathrm{tr}(V_{j_n}\cdots V_{j_1}\rho V_{j_1}^*\cdots V_{j_n}^*)$$

for every cylinder set $C_{j_1,\dots,j_n}$ with ρ a positive semi definite matrix of trace 1.

This class of probability measures can be shown to be weakly dense in the set of shift invariant probability measures but it is distinct from the class of weak Gibbs measures. One important object studied on these measures is their Rényi entropy

$$s_q(\mathbb{P}) = \lim_{n\to\infty}\frac{1}{n}\ln\sum_{j_1,\dots,j_n\in\{1,\dots,\ell\}}\mathbb{P}(C_{j_1,\dots,j_n})^q.$$

It is related to the multi-fractal spectrum of $\mathbb{P}$[1]. In the context of statistical mechanics it can be interpreted as the pressure of some potential with q playing the part of an inverse temperature. One of the main question concerning this Rényi entropy is the regularity of $q\mapsto s_q(\mathbb{P})$. It is notably related to existence of phase transitions in statistical mechanics models.

In [2] it has been proved that if the matrices $\{V_j\}_j$ are invertible and strongly irreducible then $q\mapsto s_q(\mathbb{P})$ is analytic on $(0,\infty)$. Some examples can be shown to be non C^∞ at some points in $(-\infty,0]$. The proof is based on the study of a Markov chain corresponding to $q=0$.

Recently, in [3], with some collaborators I studied a similar Markov chain but for $q=1$. We prove uniqueness of the invariant measure of the Markov chain with weaker assumptions than in [2]. Namely we do not assume the matrices are invertible and we only need irreducibility instead of strong irreducibility.

The goal of this internship would be to study the proof of [2] and to adapt it to the proof of [3] to generalize the proof of analycity of $q\mapsto s_q(\mathbb{P})$ on the open positive half line. Alternatively the interested student could study some examples (some of them related to number theory) that are expected to be non C^∞ at some point in $(-\infty,0]$.

This work could eventually be continued during a Ph.D. Thesis in probability.

References

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