MASTER LEVEL INTERNSHIP PROPOSAL:
ON A CLASS OF MEASURES ON PROJECTIVE SPACES

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In quantum physics and directional statistics, in any situation where the data at hand is a direction rather than a vector, some modification of Gaussian probability measure appear. The expression of the density is the same $f : x \mapsto \exp(-\langle x, Cx\rangle)$ but the reference measure is not Lebesgues measure anymore, but the uniform measure on the sphere $\mathrm{d}\nu$:

$$\mathrm{d}\mu(x) = e^{-\langle x,Cx\rangle}\mathrm{d}\nu(x).$$

These measures are called Bingham measures (see [2]).

Often the parameter C of the density is not accessible and only the covariance ρ of x is known. Actually, Bingham distributions are the maximal entropy ones over the sphere given the covariance ρ. Meaning that they encode the knowledge about x when only its covariance is known. Contrary to Gaussian measures, no simple relation like $C = \rho^{-1}$ is known.

The aim of this internship is to study these measures, starting from low dimension spheres where explicit computations can be carried out. Depending on the student taste, the internship could focus on the relationship between C and ρ or the behavior of the measure in different limiting regimes for C, ρ or the sphere dimension. In this last limit, under acceptable assumptions, one expects a concentration of measure phenomenon should appear, as for the uniform measure over the sphere.

The interested student could also study and compare similar measures on the sphere such as the Gaussian Adapted Projected measures recently introduced with motivations from quantum physics and quantum information (see [1]).

References

[1] S. Goldstein, J. L. Lebowitz, C. Mastrodonato, R. Tumulka, and N. Zanghì. Universal probability distribution for the wave function of a quantum system entangled with its environment. *Communications in Mathematical Physics*, 342(3):965–988, 2016.

[2] K. V. Mardia and P. E. Jupp. *Directional statistics*, volume 494. John Wiley & Sons, 2009.

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